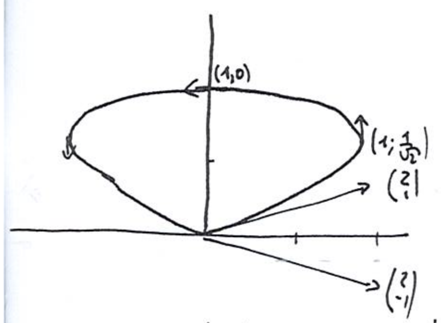
L3 GC contrôle S6 8 Avril 2016.

Exo 1 : $\Phi(x,y,z) = x^2 + 2xz + y^2 + 3z^3$ $\quad \operatorname{grad}\Phi = \begin{pmatrix} 2x+2z \\ 2y \\ 2x+9z^2 \end{pmatrix}$ $\quad \vec{n} = \operatorname{grad}\Phi(1,2,-1) = \begin{pmatrix} 0 \\ 4 \\ 11 \end{pmatrix}$

L'équation du plan tangent en A est $0.(x-1) + 4(y-2) + 11(z+1) = 0$ qui s'écrit encore $4y + 11z + 3 = 0$.

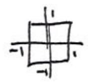
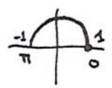
Exo 2 : $\varphi'(t) = (2\cos^2 t - 2\sin^2 t, \cos t) = (2(\cos t - \sin t)(\cos t + \sin t), \cos t)$.

$\varphi(t) = (x(t), y(t))$



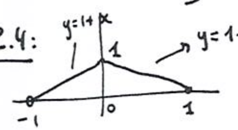
	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$		$\frac{3\pi}{4}$		π
$x'(t)$	2	$+$	0	$-$		$-$	0	$+$	2
$y'(t)$	1	$+$	$\frac{1}{\sqrt{2}}$	$+$	0	$-$	$-\frac{1}{\sqrt{2}}$	$-$	-1
$x(t)$	0	$\nearrow$	1	$\searrow$	0	$\searrow$	-1	$\nearrow$	0
$y(t)$	0	$\nearrow$	$\frac{1}{\sqrt{2}}$	$\nearrow$	1	$\searrow$	$\frac{1}{\sqrt{2}}$	$\searrow$	0

Exo 2.2 : $I_1 \overset{\text{Fubini}}{=} \int_{-1}^{1}\int_{-1}^{1} y^2 \,dx\,dy = \int_{-1}^{1} 2y^2\,dy = \frac{2}{3}\left[y^3\right]_{-1}^{1} = \frac{4}{3}$.

Exo 2.3 : $J_1 = \int\int_D x^2+y^2 \,dx\,dy \underset{\text{de var.}}{\overset{\text{chgt}}{=}} \iint_{[0;\sqrt{2}]\times[0,\pi]} r^2 \cdot r\,dr\,d\theta \overset{\text{Fubini}}{=} \int_0^{\sqrt{2}}\int_0^{\pi} r^3\,d\theta\,dr = \pi\,\frac{\sqrt{2}^4}{4} = \pi$

Exo 2.4 :



$I_3 = \iint_T y^2\,dx\,dy \overset{\text{Fubini}}{=} \int_0^1 \left(\int_{y-1}^{1-y} y^2 \quad dx\right) dy$

$= \int_0^1 2y^2(1-y)\,dy = 2\int_0^1 y^2 - y^3\,dy = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$.

Exo 2.5 : $I_4 = \iint_L y^2\,dx\,dy = \iint_L \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$ avec par exemple $Q = y^2 x$ et $P = 0$.

$\overset{\text{G.R.}}{=} \int_{\partial L^+} xy^2\,dy = \int_0^{\pi} \underbrace{2\sin t\cos t}_{x}\ \underbrace{(\sin t)^2}_{y^2}\ \underbrace{\cos t\,dt}_{dy} = \int_0^{\pi} 2\sin^3 t\,\cos^2 t\,dt = \frac{8}{15}$.

Exo 3.1 : ED linéaire d'ordre 1.

(E_1') : $y'(t) + 2t\,y(t) = 0$ $\qquad y(t) = K e^{-t^2}$ $\qquad \left(\frac{y'}{y} = -2t \,;\ \ln|y| = -t^2 + C \,;\ y = e^c e^{-t^2}\right)$

(E_1) Méthode de variation de la cte. $\quad y(t) = K(t)\,e^{-t^2} = K(t)\,z(t)$

y sol de $(E_1) \iff K'z + \underbrace{Kz' + 2tKz}_{=0 \text{ car } z \text{ sol de } (E_1')} = t \iff K' = te^{t^2} \iff K = \frac{1}{2}e^{t^2} + K_1$

Les solutions de (E_1) sont donc les fonctions $y(t) = \left(\frac{1}{2}e^{t^2} + K_1\right)e^{-t^2} = \frac{1}{2} + K_1 e^{-t^2}$.

Exo 3.2 : E.D. linéaire d'ordre 2 à coef constant.

(E_2') : $2y'' + 7y' - 4y = 0$ $\qquad P(X) = 2X^2 + 7X - 4 = 2(X - \frac{1}{2})(X+4)$ $\qquad \Delta = 7^2 + 2\times4\times4 = 49 + 32 = 81 = 9^2$

$r = \frac{-7 \pm 9}{4}$

Exo 3.2 (suite)

Solution de (E_2') : $y(t) = K_1 e^{\frac{1}{2}t} + K_2 e^{-4t}$

Pour (E_2) on cherche 1 sol. particulière de la forme $(at+b)e^{2t} = y(t)$

$y'(t) = (2at + 2b + a)e^{2t}$ $\qquad y''(t) = (4at + 4b + 2a + 2a)e^{2t}$

$2y'' + 7y' - 4y = \left[(8a + 14a - 4a)t + (8b + 8a + 14b + 7a - 4b)\right] e^{2t}$

$= \left(18a\,t + (15a + 18b)\right)e^{2t}$.

y est sol ssi $18a = 18$ et $15a + 18b = 0$ donc $a = 1$ et $b = -\frac{15}{18}$.

Les sol. de (E_2) sont $y(t) = \left(t - \frac{15}{18}\right)e^{2t} + K_1 e^{t/2} + K_2 e^{-4t}$

Exo 3.3 : Equation différentielle non linéaire.

$(E_3) \iff y'(t) = -y^2 + y + 2 \iff \frac{y'}{y^2 - y - 2} = -1 \iff \frac{dy}{y^2-y-2} = -dx$.

$\frac{1}{y^2-y-2} = \frac{1}{(y+1)(y-2)} = \frac{-1/3}{y+1} + \frac{1/3}{y-2}$

En intégrant on obtient $\frac{1}{3}\ln\left|\frac{y+1}{y-2}\right| = x + C$ donc $\ln\left|\frac{y+1}{y-2}\right| = e^{3c}e^{3x}$

D'où $y + 1 = (y-2)Ke^{3x}$ $\quad (1 - Ke^{3x})y = -2Ke^{3x} - 1$ $\quad$ D'où les sol de (E_3) : $y(x) = \frac{2Ke^{3x} + 1}{Ke^{3x} - 1}$.

Exercice 4 :

4.1 : $\Psi(x+t; y+2t) = e^{\frac{1}{3}(x+t, y+2t)} = e^{\frac{1}{3}(x+y)}e^t = e^t\,\Psi(x;y)$ donc Ψ a la pté (P).

4.2 : On pose $F(X;Y) = \Phi(x,y)$ avec $X = x$ et $Y = y + ax$.

$\Phi(x,y) = F(x; y+ax)$ $\qquad \frac{\partial\Phi}{\partial x} = \frac{\partial F}{\partial X} + a\frac{\partial F}{\partial Y}$ $\qquad \frac{\partial\Phi}{\partial y} = \frac{\partial F}{\partial Y}$

$\frac{\partial\Phi}{\partial x} + 2\frac{\partial\Phi}{\partial y} = \frac{\partial F}{\partial X} + (a+2)\frac{\partial F}{\partial Y}$ $\qquad$ Posons $a = -2$

Φ est sol de (E) ssi $\frac{\partial F}{\partial X} = F$ cad $F(X;Y) = K(Y)\,e^X$

Les sol. de (E) sont les fonctions $\Phi(x,y) = K(y+2x)\,e^x$.

4.3 : Si $\Phi(x,y) = K(y-2x)\,e^x$

$\Phi(x+t, y+2t) = K(y + 2t - 2x - 2t)\,e^{x+t} = K(y-2x)\,e^x e^t = e^t\,\Phi(x;y)$.

4.4 : Dérivons la pté (P) par rapport à t on obtient.

$\frac{\partial\Phi}{\partial x}(x+t, y+2t) + 2\frac{\partial\Phi}{\partial y}(x+t; y+2t) = e^t\,\Phi(x,y)$.

Pour $t = 0$ cela donne $\frac{\partial\Phi}{\partial x}(x,y) + 2\frac{\partial\Phi}{\partial y}(x;y) = \Phi(x,y)$

4.5 : Les champs scalaires ayant la pté (P) sont les fct $\Phi(x,y) = K(y-2x)\,e^x$.

4.6 : $K(-a-2a)e^a = a^2$ $\qquad K(-3a) = a^2 e^{-a}$ $\qquad$ donc $K(x) = \left(\frac{x}{3}\right)^2 e^{x/3}$

Donc $\Phi(x,y) = \frac{(y-2x)^2}{9}\,e^{\frac{y-2x}{3}}\,e^x = \frac{(y-2x)^2}{9}\,e^{\frac{x+y}{3}}$.