Examen de Mathématiques 10 Avril 2015 L3SPI

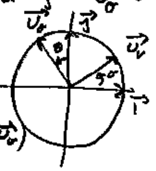
Exercice 1 :

1.1 $\mathrm{Div}(A) = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} = \dfrac{2x}{\sqrt{x^2+y^2}} - \dfrac{1}{2}\dfrac{2x(x^2+y^2)}{(x^2+y^2)^{3/2}} + \dfrac{2x}{\sqrt{x^2+y^2}} - \dfrac{1}{2}\dfrac{2xy\,(2y)}{(x^2+y^2)^{3/2}}$

$= \dfrac{4x}{\sqrt{x^2+y^2}} - \dfrac{x^3 + xy^2 + 2xy^2}{(x^2+y^2)^{3/2}} = \dfrac{4x}{\sqrt{x^2+y^2}} - \dfrac{x\,(x^2+y^2)}{(x^2+y^2)^{3/2}} = \dfrac{3x}{\sqrt{x^2+y^2}}$

1.2 : $r = \sqrt{x^2+y^2}$ $\tan\theta = \dfrac{y}{x}$ $y = r\sin\theta$ $x = r\cos\theta$ $\vec{U_r} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$ $\vec{U_\theta} = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}$

$\begin{cases} \vec{U_r} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \\ \vec{U_\theta} = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} \end{cases}$ $\begin{cases} \vec{\imath} = \cos(\theta)\vec{U_r} + \sin(-\theta)\vec{U_\theta} = \cos\theta\,\vec{U_r} - \sin\theta\,\vec{U_\theta} \\ \vec{\jmath} = -\sin(-\theta)\vec{U_r} + \cos(-\theta)\vec{U_\theta} = \sin\theta\,\vec{U_r} + \cos\theta\,\vec{U_\theta} \end{cases}$

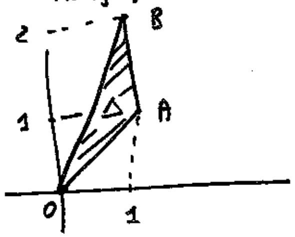


$\tilde{A} = \dfrac{r^2\cos^2\theta - r^2\sin^2\theta}{r}\left(\cos\theta\,\vec{U_r} - \sin\theta\,\vec{U_\theta}\right) + \dfrac{2r^2\cos\theta\sin\theta}{r}\left(\sin\theta\,\vec{U_r} + \cos\theta\,\vec{U_\theta}\right)$

$= \left(r\cos^3\theta - r\sin^2\cos\theta + 2r\cos\theta\sin^2\theta\right)\vec{U_r} + \left(-r\cos^2\theta\sin\theta + r\sin^3\theta + 2r\cos^2\theta\sin\theta\right)\vec{U_\theta}$

$= r\cos\theta\,(\cos^2\theta + \sin^2\theta)\,\vec{U_r} + r\sin\theta\,(\cos^2\theta + \sin^2\theta)\,\vec{U_\theta} = \underbrace{r\cos\theta}_{A_r}\,\vec{U_r} + \underbrace{r\sin\theta}_{A_\theta}\,\vec{U_r}$

$\mathrm{div}(\tilde{A}) = \cos\theta + \cos\theta + \dfrac{1}{r}\,r\cos\theta = 3\cos\theta \quad \left(= \dfrac{3x}{\sqrt{x^2+y^2}}\right).$



Exercice 2 :

2.2 : $I \overset{\text{Fubini}}{=\!=} \displaystyle\int_0^1 \left(\int_x^{2x} x^2 + y^2\,dy\right) dx$

$= \displaystyle\int_0^1 x^3 + \frac{1}{3}\left[y^3\right]_x^{2x} dx = \int_0^1 x^3 + \frac{1}{3}7x^3\,dx$

$= \dfrac{10}{3}\displaystyle\int_0^1 x^3\,dx = \frac{5}{6}.$

2.3 : Appliquons Green Riemann. $\displaystyle\int_{\partial K} P\,dx + Q\,dy = \iint_K \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$

$J = \displaystyle\iint_\Delta \frac{1}{x}\,dx\,dy = \int_0^1 \int_x^{2x} \frac{1}{x}\,dy\,dx = \int_0^1 1\,dx = 1.$

Exercice 3.

(E1) C'est une E.D.O. à variables separable

$yy' = x^3$ $\quad y\,dy = x^3 dx \quad \dfrac{y^2}{2} = \dfrac{x^4}{4} + c \;\Big|\; y = \pm\sqrt{\dfrac{2x^4 + 8c}{4}}$

Exercice 3 (Suite)

E.D.L. à coeff constant d'ordre 3.

(E_2') : $y''' - 3y'' + 7y' - 5y = 0$ Equation caracteristique $r^3 - 3r^2 + 7r - 5 = 0$

or $r^3 - 3r^2 + 7r - 5 = (r-1)(r^2 - 2r + 5)$ $\quad \Delta = 4 - 20 = -16 = (4i)^2 \quad r = 1 \pm 2i$

$= (r-1)(r-1-2i)(r-1+2i)$

Sol. de (E_2') : $y(x) = K_1 e^x + e^x\left(K_2\cos 2x + K_3\sin 2x\right).$

On cherche une solution particulière de (E_2) sous la forme

$y(x) = K e^{-x}$ $\quad y$ sol. $\Leftrightarrow -K - 3K - 7K - 5K = 1 \Leftrightarrow K = \dfrac{-1}{16}$

Sol. de (E_2) $\quad y(x) = -\dfrac{e^{-x}}{16} + e^x\left(K_1 + K_2\cos 2x + K_3\sin 2x\right).$

Exercice 4 :

$\begin{cases} X = x \\ Y = ax + y \end{cases}$ $\quad f(x,y) = F(X,Y) = F(x, ax+y).$ $\quad \dfrac{\partial f}{\partial x} = \dfrac{\partial F}{\partial X} + a\dfrac{\partial F}{\partial Y}$ $\quad \dfrac{\partial f}{\partial y} = \dfrac{\partial F}{\partial Y}$

(E_1) $\quad \dfrac{\partial f}{\partial x} + 5\dfrac{\partial f}{\partial y} = 3f + 1$

$\dfrac{\partial F}{\partial X} + a\dfrac{\partial F}{\partial Y} + 5\dfrac{\partial F}{\partial Y} = 3F + 1$ Prenons $a = -5$

$\dfrac{\partial F}{\partial X} = 3F + 1$ on intègre l'équation à Y fixé $F(X,Y) = K(Y)e^{3X} - \dfrac{1}{3}.$

on obtient donc $f(x,y) = K(y - 5x)\,e^{3x} - \dfrac{1}{3}.$

$f(x;2x) = K(-3x)\,e^{3x} - \dfrac{1}{3} = 0$ $\quad$ Donc $K(-3x) = \dfrac{1}{3}e^{-3x}$ donc $K(X) = \dfrac{1}{3}e^X$

Finalement $f(x;y) = \dfrac{1}{3}\left[e^{y-5x}\,e^{3x} - 1\right] = \dfrac{1}{3}\left[e^{y-2x} - 1\right].$

(E_2) : $\dfrac{\partial F}{\partial X} + a\dfrac{\partial F}{\partial Y} - \dfrac{\partial F}{\partial Y} = \dfrac{F}{X(X+2)}$ $\quad$ Prenons $a = 1$.

$\dfrac{\partial F}{\partial X} = \dfrac{F}{X(X+2)}$ $\quad$ On intègre l'équation à Y fixé.

(Elle s'intègre comme $y' - \dfrac{1}{t(t+2)}\,y = 0$ E.D.L. à coef variable).

$\dfrac{\frac{\partial F}{\partial X}}{F} = \dfrac{1}{X(X+2)} = \dfrac{1/2}{X} - \dfrac{1/2}{X+2}$ $\qquad \ln|F(X;Y)| = \dfrac{1}{2}\ln|X| - \dfrac{1}{2}\ln|X+2| + k$

$F(X;Y) = \sqrt{\left|\dfrac{X}{X+2}\right|}\;C(Y)$ $\qquad f(x,y) = \sqrt{\left|\dfrac{x}{x+2}\right|}\;C(x+y).$

Verification : $\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y} = \dfrac{2}{(x+2)^2}\,\dfrac{1}{2\sqrt{\frac{x}{x+2}}}\,C + \sqrt{\dfrac{x}{x+2}}\,C' - \sqrt{\dfrac{x}{x+2}}\,C' = \dfrac{f(x,y)}{x(x+2)}$