Examen L2GC / Pre ing 2 — 10 Mai 2022

Exo 1 :

(E_1) $5y''+8y'+5y=1$ ED linéaire d'ordre 2 à coef cst.
$P(X)=5X^2+8X+5$ $\quad \Delta = 64-100 = -36 = (6i)^2$ $\quad r = \frac{-8\pm 6i}{10} = \frac{-4}{5} \pm \frac{3i}{5}$.
les sol. de (E_1') sont $y(x) = e^{-\frac{4}{5}x}\left(K_1 \cos \frac{3}{5}x + K_2 \sin \frac{3}{5}x\right)$
$y=\frac{1}{5}$ est 1 solution particulière. Sol de (E_1) : $y(x) = \frac{1}{5} + e^{-\frac{4}{5}x}\left(K_1 \cos(\frac{3}{5}x) + K_2 \sin(\frac{3}{5}x)\right)$

$(E_2) \Leftrightarrow \frac{2y}{y^2+1} y' = \frac{-1}{1+x^2}$ ED non linéaire à variables séparables.

$\int \frac{2y\,dy}{y^2+1} = -\int \frac{dx}{1+x^2}$ soit $\ln(y^2+1) = -\operatorname{Arctan} x + K$
en 0 : $\ln(2) = -\operatorname{Arctan} 0 + K = K$ donc $K = \ln 2$.

$\ln \frac{y^2+1}{2} = -\operatorname{Arctan} x$ donc $y^2 = 2\exp(-\arctan x) - 1$ donc $y(x) = \sqrt{2\exp(-\arctan x)-1}$

(E_3) $\varphi(x,y) = \xi(u,v) = \xi(x+ay\,;\,y)$ $\quad \frac{\partial\varphi}{\partial x} = \frac{\partial \xi}{\partial u}$ $\quad \frac{\partial\varphi}{\partial y} = a\frac{\partial\xi}{\partial u} + \frac{\partial\xi}{\partial v}$
φ solution $\Leftrightarrow \frac{\partial\xi}{\partial u} - 2\left(a\frac{\partial\xi}{\partial u} + \frac{\partial\xi}{\partial v}\right) = -2\xi$ Posons $a=\frac{1}{2}$. On a $\frac{\partial\xi}{\partial v} = \xi$
on intègre en v à u fixé on obtient $\xi(u;v) = K(u)\,e^{v}$ $\quad \varphi(x,y) = K(x+\frac{1}{2}y)\,e^{y}$
$\varphi(t,2t) = K(2t)\,e^{2t} = 1$ Donc $K(s) = e^{-s}$ donc $\varphi(x,y) = e^{-x+\frac{1}{2}y}$

Exercice 2 : $4(1)e^{2-2} - 2.4 + 2+2 = 4-8+4=0$. $\quad \Phi(x,y,z) = 4(z-1)e^{2y-x} - 2z^2 + xy + 2$

q2 . $\vec{n} = \nabla\Phi = \begin{pmatrix} -4(z-1)e^{2y-x} + y \\ 8(z-1)e^{2y-1} + x \\ 4e^{2y-1} - 4z \end{pmatrix} \xrightarrow{\text{en } G} \begin{pmatrix} -4+1 \\ 8+2 \\ 4-8 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ -4 \end{pmatrix}$

q3 . $\vec{AB}$ et $\vec{AC}$ sont-ils colinéaires ? $\quad \vec{AB} \wedge \vec{AC} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ oui donc ABC alignés.

q.4 . un point de la droite AB s'écrit $A + t\vec{AB} = (0-2t\,;0-t\,;1-t)$

$\Phi(-2t\,;-t\,;1-t) = 4(-t)\,e^{-2t+2t} - 2(1-t)^2 + (-2t)(-t) + 2$
$= -4t - 2 + 4t - 2t^2 + 2t^2 + 2 = 0$ donc $P_t (-2t\,;-t\,;1-t) \in \Sigma$

q.5 . $-3(x-2) + 10(y-1) \quad -4(z-2) = 0$ car $\vec{CM} \perp \vec{n}$ ssi $M \in T$.

q.6 : $-3(-2t-2) + 10(-t-1) \quad -4(1-t-2) = 6t+6-10t-10+4+4t = 0$ donc $P_t \in T$.

Exercice 3 : $[(x,y,z).\nabla]\,(x+2y+3xyz) = x(1+3yz) + y(2+3xz) + z(3xy) = x+2y+9xyz$

$[(1,y,z^2).\nabla]\begin{pmatrix} xyz \\ 1 \\ e^{2x} \end{pmatrix} = \begin{pmatrix} yz + xyz + xyz^2 \\ 0 \\ 2e^{2x} \end{pmatrix}$

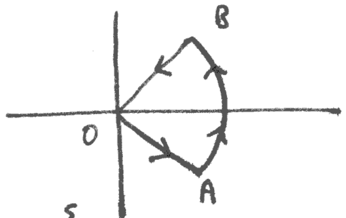
3.3 $\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} \Psi_2\Phi_3 - \Phi_2\Psi_3 \\ \Psi_3\Phi_1 - \Phi_3\Psi_1 \\ \Psi_1\Phi_2 - \Phi_1\Psi_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}(\Psi_1\Phi_2 - \Phi_1\Psi_2) - \frac{\partial}{\partial z}(\Psi_3\Phi_1 - \Phi_3\Psi_1) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \Pi_1 \\ \\ \\ \end{pmatrix}$

$\Pi_1 = \underset{e}{\frac{\partial\Psi_1}{\partial y}}\Phi_2 + \Psi_1\underset{a}{\frac{\partial\Phi_2}{\partial y}} - \underset{g}{\frac{\partial\Phi_1}{\partial y}}\Psi_2 - \Phi_1\underset{c}{\frac{\partial\Psi_2}{\partial y}} - \underset{d}{\frac{\partial\Psi_3}{\partial z}}\Phi_1 - \Psi_3\underset{h}{\frac{\partial\Phi_1}{\partial z}} + \Phi_3\underset{f}{\frac{\partial\Psi_1}{\partial z}} + \underset{b}{\frac{\partial\Phi_3}{\partial z}}\Psi_1$.

3.4 $\Pi_2 = \Big(\underset{X_1}{\frac{\partial\Phi_1}{\partial x}} + \underset{a}{\frac{\partial\Phi_2}{\partial y}} + \underset{b}{\frac{\partial\Phi_3}{\partial z}}\Big)\Psi_1 - \Big(\underset{X_2}{\frac{\partial\Psi_1}{\partial x}} + \underset{c}{\frac{\partial\Psi_2}{\partial y}} + \underset{d}{\frac{\partial\Psi_3}{\partial z}}\Big)\Phi_1 + \Phi_1\underset{X_2}{\frac{\partial\Psi_1}{\partial x}} + \Phi_2\underset{e}{\frac{\partial\Psi_1}{\partial y}} + \Phi_3\underset{f}{\frac{\partial\Psi_1}{\partial z}} - \Psi_1\underset{X_1}{\frac{\partial\Phi_1}{\partial x}} - \Psi_2\underset{g}{\frac{\partial\Phi_1}{\partial y}} - \Psi_3\underset{h}{\frac{\partial\Phi_1}{\partial z}}$

Exercice 4 :

$$I = \iint_{\substack{0 \le r \le \sqrt{2} \\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}}} r^3 \cos\theta \sin^2\theta \, r\,dr\, d\theta.$$



$$= \int_0^{\sqrt{2}} r^4\, dr \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\theta \sin^2\theta \, d\theta = \left[\frac{r^5}{5}\right]_0^{\sqrt{2}} \left[\frac{1}{3}\sin^3\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\sqrt{2}^5}{5} \cdot \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^2 2 = \frac{4\sqrt{2}}{15}.$$

Green Riemann

$$J = \int_{\partial C^+} y\, dx + x^2 y^2\, dy = \iint_C 2xy^2 - 1 \, dx\,dy = 2I - \text{aire}(C) = \frac{8\sqrt{2}}{15} - \frac{1}{4}\pi.2$$
$$= \frac{8\sqrt{2}}{15} - \frac{\pi}{2}.$$

Exo 5 : 5.1 non sauf si $b=0$.

5.2. $z(x) = \frac{1}{y(x)}$ donc $y(x) = \frac{1}{z(x)}$ donc $y'(x) = \frac{-z'(x)}{z(x)^2}$.

y sol de $(E) \iff y' + ay + by^2 = 0 \iff \frac{-z'}{z^2} + a\frac{1}{z} + b\frac{1}{z^2} = 0 \iff \underbrace{-z' + az + b = 0}_{(F)\text{ ED linéaire d'ordre 1}}$.

5.3a : $(E) \iff y' + \underbrace{\frac{1}{(1+x)^2}}_{a(x)} y - \underbrace{\frac{e^{\frac{1}{1+x}}}{(1+x)^3}}_{b(x)} y^2 = 0.$

$(F) : -z' + \frac{1}{(1+x)^2} z - \frac{e^{\frac{1}{1+x}}}{(1+x)^3} = 0.$

5.3b : $(F') \quad -z' + \frac{1}{(1+x)^2} z = 0 \iff \frac{z'}{z} = \frac{1}{(1+x)^2} \iff \ln|z| = \frac{-1}{1+x} + C \iff z(x) = K \underbrace{e^{\frac{-1}{1+x}}}_{z_0(x)}$

5.3c : on cherche 1 sol. particulière de (F) de la forme

$z(x) = K(x) z_0(x)$ avec z_0 sol. de (F').

z sol. de (F) ssi $-(Kz_0' + K'z_0) + \frac{1}{(1+x)^2} z Kz_0 - \frac{e^{\frac{1}{1+x}}}{(1+x)^3} = 0$ or $-z_0' + \frac{1}{(1+x)^2} z_0 = 0$

ssi $-K' e^{-\frac{1}{1+x}} - \frac{e^{\frac{1}{1+x}}}{(1+x)^3} = 0$

ssi $K'(x) = \frac{e^{\frac{2}{1+x}}}{(1+x)^3}$

$K(x) = \int \frac{e^{\frac{2}{1+x}}}{(1+x)^3} dx$ effectuons le changt de variable $t = \frac{1}{1+x}$ $dt = \frac{-dx}{(1+x)^2}$

$= -\int t e^{2t} dt = -\left[\frac{1}{2} t e^{2t}\right] + \int \frac{1}{2} e^{2t} = -\frac{1}{2} t e^{2t} + \frac{1}{4} e^{2t} = \left(-\frac{1}{2}\frac{1}{1+x} + \frac{1}{4}\right) e^{\frac{2}{1+x}}$

Donc solution de (F) : $z(x) = \left(-\frac{1}{2(1+x)} + \frac{1}{4}\right) e^{\frac{1}{1+x}} + k e^{-\frac{1}{1+x}}$

5.3.d Donc sol. de (E) $y(x) = \frac{1}{\left(-\frac{1}{2(1+x)} + \frac{1}{4}\right) e^{\frac{1}{1+x}} + k e^{-\frac{1}{1+x}}}$.