Université de Cergy-Pontoise 2010/2011
Licence L2 MPI

Examen de Séries (M3S)
(durée 3 heures - Documents, calculators, mobile phones are forbidden)

Question on the lecture.–
1) Recall the alternating series theorem and give an example of application.
2) Recall the criteria giving the nature of a Riemann series.

Exercice 1.– Study the nature of the numerical series whose general term is given below :

$$\left(\frac{2n+1}{3n+4}\right)^n, \quad \frac{(\cos n)^3}{n\sqrt{n}}, \quad \frac{1}{n^2+1}\big((-1)^n n+2\big)$$

Exercice 2.– Consider the sequence of functions defined on $I=[0,1]$ by $f_n(x)=\dfrac{ne^x}{n+x}$.
1) Show that this sequence of functions converges simply on I to a function f to be determined.
2) Show that $\forall x \in I$, we have $\left|\dfrac{xe^x}{n+x}\right| \leq \dfrac{e}{n}$.
3) Study the uniform convergence of this sequence of functions on I.
4) Calculate the value of the following limit $lim_{n\to+\infty}\displaystyle\int_0^1 \frac{ne^x}{n+x}dx$ (give a detailed justification of your reasoning) .

Exercice 3.– For $n \geq 1$ let $u_n : \mathbb{R} \to \mathbb{R}$ be the function defined by $u_n(x)=\dfrac{e^{-\sqrt{n}x}}{1+n^2}$.
1) Show that the series of functions $\sum_{n\geq 1} u_n(x)$ converges simply on $\mathbb{R}^+$. Does it converge simply on $\mathbb{R}$?
2) Study the normal convergence on $\mathbb{R}^+$.
3) We denote by $S(x)=\sum_{n=1}^{+\infty} u_n(x)$ the sum of the series on $\mathbb{R}^+$.
a) Is S well defined on $\mathbb{R}^+$?
b) Is S continuous on $\mathbb{R}^+$?
c) Is S a C^1-class function on $\mathbb{R}^+$?

Exercice 4.–
A– Cours :
A1) Recall the differentiation theorem of the sum of a power series.
A2) Show that the function $-\ln(1-x)$ is developpable into a power series on the interval $]-1,1[$ and that this development is : $-\ln(1-x)=\sum_{n=1}^{+\infty}\frac{x^n}{n}$.

B– For $n \geq 1$ and $x \in \mathbb{R}$ put $u_n(x)=\frac{x^n}{n(n+1)}$; we propose to study of the series of functions $\sum_{n\geq 1} u_n(x)$ and the properties of its sum $S(x)$.
1) Calculate R the convergence radius of the series ?
2) Study the convergence of the series for $x=R$ and for $x=-R$, and deduce D the domain of simple convergence of the series.
3) Show that its sum denoted $S(x)$ is continuous on D.
4) a) Decompose into simple eléments the rational fraction $\frac{1}{t(t+1)}$.
b) By using the development given in A2) and the decomposition obtained in 4)a-, deduce the expression of $S(x)$ on $]-R,0[\cup]0,R[$, next give the expression of $S(x)$ on $]-R,R[$.
5) By using the previous results and by giving the appropriate justifications calculate the sums of the

two following numerical series :

$$\sum_{n\geq 1} \frac{1}{n(n+1)}, \qquad \sum_{n\geq 1} \frac{(-1)^n}{n(n+1)}.$$