Examen Algèbre 3

12 Exercice 1 :

1 1) $A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

2 2) $P(X) = -(X-1)(X+2)^2$

1 3) $Sp(A) = \{1; -2\}$.

4 4) $E_1 = \mathrm{Vect}\left((1,1,1)\right)$ $\quad \dim(E_1) = 1$

$E_{-2} = \mathrm{Vect}\left((-1,1,0), (-1,0,1)\right)$ $\quad \dim(E_{-2}) = 2$

1 5) $\dim(\mathbb{R}^3) = \dim(E_1) + \dim(E_{-2})$

donc f est diagonalisable

2 6) $\widetilde{B} = \left((1,1,1), (-1,1,0), (-1,0,1)\right)$

$\Pi = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

1 7) $D = \Pi^{-1} A \Pi$

5 Exercice 2 :

1 1) $A^2 = \begin{pmatrix} -2 & 1 & 1 \\ -3 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix}$

2 2) $P(X) = -X^3 + X$

2 3) $P(A) = O_3 \Leftrightarrow A^3 = A$ et $A^6 = A^2$.

8 Exercice 3 :

1 1) $\forall n \in \mathbb{N}, \quad \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}}_{A} \begin{pmatrix} u_n \\ v_n \end{pmatrix}$

1 (si réc) 2) $\forall n \in \mathbb{N}, \quad \begin{pmatrix} u_n \\ v_n \end{pmatrix} = A^n \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$. par réc.

2 3) $P(X) = X^2 - 5X + 4 = (X-1)(X-4)$

$Sp(A) = \{1, 4\}$

$E_1 = \mathrm{Vect}\left((1,-2)\right)$ $\qquad P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$E_4 = \mathrm{Vect}\left((1,1)\right)$

1 4) $P^{-1} = \begin{pmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{pmatrix}$

2 5) $\forall n \in \mathbb{N}, \quad \underbrace{P^{-1}\begin{pmatrix} u_n \\ v_n \end{pmatrix}}_{\begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix}} = D^n \underbrace{P^{-1}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}}_{\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix}}, \quad \begin{pmatrix} \tilde{u}_n \\ \tilde{v}_n \end{pmatrix} = D^n \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix}$

$\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1 6) $\tilde{u}_6 = 1 \qquad u_6 = -1 + 4^6$

$\tilde{v}_6 = 4^6 \qquad v_6 = -2 + 4^6$.

Examen algèbre 3

(12) **Exercice 1 :** $f(x,y,z) = (-x+y+z,\ x-y+z,\ x+y-z)$

1 — 1) $A = \mathrm{Mat}_{B,B}(f) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

2) $P(X) = \det(A - XI) = \begin{vmatrix} -1-X & 1 & 1 \\ 1 & -1-X & 1 \\ 1 & 1 & -1-X \end{vmatrix}$

$L_2 \leftarrow L_2 - L_1$ $= \begin{vmatrix} -1-X & 1 & 1 \\ 2+X & -2-X & 0 \\ 1 & 1 & -1-X \end{vmatrix} = (2+X)\begin{vmatrix} -1-X & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1-X \end{vmatrix}$

$C_2 \leftarrow C_2 + C_1$ $= (2+X)\begin{vmatrix} -1-X & -X & 1 \\ 1 & 0 & 0 \\ 1 & 2 & -1-X \end{vmatrix} = (2+X)\times(-1)\times\begin{vmatrix} -X & 1 \\ 2 & -1-X \end{vmatrix}$

2 — $= -(2+X)(X + X^2 - 2) = -(2+X)(X-1)(X+2) = -(X-1)(X+2)^2$

1 — 3) $P(X) = 0 \iff X = 1$ ou $X = -2$. $\mathrm{sp}(A) = \{1 ; -2\}$.

1 — 4) Recherche de $E_1 = \ker(A - I)$

Soit $u = (x,y,z) \in \mathbb{R}^3$.

$u \in E_1 \iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2x+y+z=0 \\ x-2y+z=0 \\ x+y-2z=0 \end{cases}$

2 — $\begin{matrix} L_1 \leftrightarrow L_3 \\ L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 + 2L_1 \end{matrix} \iff \begin{cases} x+y-2z=0 \\ -3y+3z=0 \\ 3y-3z=0 \end{cases} \iff \begin{cases} x=z \\ y=z \\ z=z \end{cases} \iff u = z(1,1,1)$

$E_1 = \mathrm{Vect}((1,1,1))$.

$\dim E_1 = 1$.

Recherche de $E_2 = \ker(A + 2I)$

Soit $u = (x,y,z) \in \mathbb{R}^3$.

$u \in E_2 \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x+y+z=0 \\ y=y \\ z=z \end{cases}$

2 — $\iff \begin{cases} x=-y-z \\ y=y \\ z=z \end{cases} \iff u = y(-1,1,0) + z(-1,0,1)$.

$E_{-2} = \mathrm{Vect}((-1,1,0),(-1,0,1))$.

$\dim E_{-2} = 2$.

1 — 5. $\dim(\mathbb{R}^3) = \dim E_1 + \dim E_{-2}$ donc f est diagonalisable.

2 — 6. $\tilde{B} = ((1,1,1), (-1,1,0), (-1,0,1))$.

$M = P_{B,\tilde{B}} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

1 — 7. $D = M^{-1} A M$

(5) **Exercice 2 :**

1 — 1) $A^2 = \begin{pmatrix} -2 & 1 & 1 \\ -3 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix}$

2 — 2) $P(X) = \det(A - XI) = \begin{vmatrix} 1-X & -1 & 0 \\ 3 & -2-X & -1 \\ 0 & -1 & 1-X \end{vmatrix}$

$= (1-X)(-2-X)(1-X) + 3(1-X) - 1\times(1-X)$.

$= (X^2 - 2X + 1)(-2-X) + 2 - 2X$

$= -X^3 - 2X^2 + 4X + 2X^2 - 2 - 2X + 2 - X$

$= -X^3 + X = X(1-X)(1+X)$ $a=-1$ $b=0$ $c=1$ $d=0$.

(1)

3) On a $P(X) = -X^3 + X$ et P est un polynôme annulateur de A donc $-A^3 + A = 0$

2 ainsi $A^3 = A$ et $\underline{A^6 = (A^3)^2 = A^2} = \begin{pmatrix} -2 & 1 & 1 \\ -3 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix}$

(8) Exercice 3 :

1) $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ $X_n = \begin{pmatrix} U_n \\ V_n \end{pmatrix}$ $X_{n+1} = \begin{pmatrix} U_{n+1} \\ V_{n+1} \end{pmatrix}$

1 $\forall n \in \mathbb{N}$, $\underline{X_{n+1} = AX_n}$

1 (si réc) 2) On démontre par récurrence que :

$\forall n \in \mathbb{N}$, $\underline{X_n = A^n X_0}$

3). $P(X) = \det(A - XI) = \begin{vmatrix} 3-X & 1 \\ 2 & 2-X \end{vmatrix} = (3-X)(2-X) - 2$

$= X^2 - 5X + 4 = (X-1)(X-4)$

- $Sp(A) = \{1, 4\}$. A est diagonalisable.
- $E_1 = \ker(A - I)$.

$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x + y = 0 \\ x = x \end{cases} \Leftrightarrow \begin{cases} x = x \\ y = -2x \end{cases}$

$E_1 = \mathrm{Vect}((1, -2))$

2
- $E_4 = \ker(A - 4I)$

$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -x + y = 0 \\ x = x \end{cases} \Leftrightarrow \begin{cases} x = x \\ y = x \end{cases}$

$E_2 = \mathrm{Vect}((1, 1))$.

$P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ $D = P^{-1}AP$

$A = PDP^{-1}$

1 4) $P^{-1} = \begin{pmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{pmatrix}$

5) $\forall n \in \mathbb{N}$, $X_n = PD^n P^{-1} X_0$

2 et $\underbrace{P^{-1}X_n}_{\begin{pmatrix} \tilde{U}_n \\ \tilde{V}_n \end{pmatrix}} = D^n \underbrace{P^{-1}X_0}_{\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix}}$ ainsi $\begin{pmatrix} \tilde{U}_n \\ \tilde{V}_n \end{pmatrix} = D^n \begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix}$.

$\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix} = P^{-1}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1 6) $\begin{pmatrix} \tilde{u}_6 \\ \tilde{v}_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4^6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4^6 \end{pmatrix}$ $\tilde{u}_6 = 1$, $\tilde{v}_6 = 4^6$

et $\begin{pmatrix} u_6 \\ v_6 \end{pmatrix} = P\begin{pmatrix} \tilde{u}_6 \\ \tilde{v}_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4^6 \end{pmatrix} = \begin{pmatrix} 1 + 4^6 \\ -2 + 4^6 \end{pmatrix}$ $u_6 = 1 + 4^6$, $v_6 = -2 + 4^6$.