E1: $\int_0^{\pi/2} t\cos(2t)\,dt = \left[t\,\frac{\sin(2t)}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin(2t)}{2}\,dt = +\left[\frac{\cos 2t}{4}\right]_0^{\pi/2} = -\frac{1}{2}.$

$\int_1^2 \frac{1}{t(1+t)}\,dt = \int_1^2 \frac{1}{t} + \frac{-1}{1+t}\,dt = \left[\ln t - \ln(1+t)\right]_1^2 = \ln 2 - \ln 3 + \ln 2 = 2\ln 2 - \ln 3.$

E2: $\frac{\partial\varphi}{\partial x} = 4x(x^2+y^2) - 2(x+y+z-3)$ $\frac{\partial\varphi}{\partial y} = 4y(x^2+y^2) - 2(x+y+z-3)$ $\frac{\partial\varphi}{\partial z} = -2(x+y+z-3)$

2: Pt critique $\begin{cases} x+y+z-3=0.\\ 4x(x^2+y^2)=0\\ 4y(x^2+y^2)=0\end{cases}$ Seul point critique $(0,0,3)$.

3: $\varphi(t,-t,3)>0$ et $\varphi(0,0,3+t)<0$ Donc φ prend des valeurs >0 et des valeurs <0 aussi près que l'on veut de $(0,0,3)$ donc pas d'extremum local en $(0,0,3)$.
Si φ avait un extremum local en un autre point ce serait un point critique.

Ex3: $\frac{\partial\varphi}{\partial x} = \frac{(x-y)e^x-(e^x-e^y)}{(x-y)^2}$ $\frac{\partial\varphi}{\partial y} = \frac{-(x-y)e^y+(e^x-e^y)}{(x-y)^2}$ en (x,y) tq $x\neq y$.

2: $\varphi(1+h,k) = \varphi(1,0) + \frac{\partial\varphi}{\partial x}(1,0)\,h + \frac{\partial\varphi}{\partial y}(1,0)\,k + o(\|h,k\|) = (e-1) + \frac{e-e+1}{1}h + \frac{-1+e-1}{1}k + o(\sqrt{h^2+k^2})$

3: Il faut que $\varphi(1+h,k) - \varphi(1,0)$ varie le moins possible $= (e-1) + h + (e-2)k + o(\sqrt{h^2+k^2})$.
donc que $h + (e-2)k = 0$ donc dans la direction $\begin{pmatrix} e-2\\ -1\end{pmatrix}$.

4: $\frac{\varphi(t+h,t)-\varphi(t,t)}{h} = \frac{\frac{e^{t+h}-e^t}{h} - e^t}{h} = e^t\,\frac{\frac{e^h-1}{h}-1}{h} = e^t\,\frac{e^h-1-h}{h^2} \xrightarrow[h\to 0]{} \frac{1}{2}e^t = \frac{\partial\varphi}{\partial x}(t,t)$

Ex4: $\varphi(x,y) = -1 + 3(x-1) + 6(y-2) + \frac{1}{2}\left[2(x-1)^2 + 6(x-1)(y-2) + 5(y-2)^2\right] + o\left((x-1)^2+(y-2)^2\right)$

2) Non car $(1,2)$ n'est pas 1 point critique

3) $f(t) = \varphi\left(e^{2t}, \frac{2}{\sqrt{1+t}}\right)$ $f'(t) = 2e^{2t}\frac{\partial\varphi}{\partial x}\left(e^{2t}, \frac{2}{\sqrt{1+t}}\right) - (1+t)^{-3/2}\frac{\partial\varphi}{\partial y}\left(e^{2t}, \frac{2}{\sqrt{1+t}}\right)$ $f(0)=-1$ $f'(0)=6-6=0.$

4) On peut calculer la dérivée seconde de f en 0, en dérivant $f'(t)$ puis en prenant $t=0$.
Sinon on peut utiliser le DL_2 $e^{2t} = 1+2t+2t^2+o(t^2)$ $\frac{2}{\sqrt{1+t}} = 2 - t + \frac{3}{4}t^2 + o(t^2)$

$f(t) = -1 + 3(2t+2t^2) + 6(-t+\frac{3}{4}t^2) + (2t+2t^2)^2 + 3(2t+2t^2)(-t+\frac{3}{4}t^2) + \frac{5}{2}(-t+\frac{3}{4}t^2)^2 + o(t^2)$
$= -1 + t^2\left(6 + \frac{9}{2} + 4 - 6 + \frac{5}{2}\right) + o(t^2) = -1 + {}'' t^2 + o(t^2) = -1 + \underbrace{t^2}_{>0}\underbrace{(11+\varepsilon(t))}_{>0 \text{ pour } t \text{ petit}}$

Donc $f(t)-f(0)>0$ pour t petit f possède un min en 0.

Exo5: $\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} f(\frac{k}{n}) - f(t)\,dt = \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} f(\frac{k}{n})\,dt + \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} f(t)\,dt = \sum_{k=0}^{n-1}\frac{1}{n}f(\frac{k}{n}) + \int_0^1 f(t)\,dt$ (Chasles).

5.2: $\left|\frac{1}{n}\sum_{k=0}^{n} f(\frac{k}{n}) - \int_0^1 f(t)\,dt\right| \leq \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}\left|f(\frac{k}{n}) - f(t)\right|dt \leq \sum_{k=0}^{n-1} K\int_{k/n}^{(k+1)/n} t - \frac{k}{n}\,dt = \frac{K}{2}\sum_{k=0}^{n-1}\left[(t-\frac{k}{n})^2\right]_{k/n}^{(k+1)/n} = \frac{K}{2}\sum_{k=0}^{n-1}\frac{1}{n^2}$

Donc $\left|\frac{1}{n}\sum_{k=0}^{n} f(\frac{k}{n}) - \int_0^1 f(t)\,dt\right| \leq \frac{K}{2}\,n\,\frac{1}{n^2} = \frac{K}{2n}$

5.3: ici $f(t) = \cos(t^2)$ $f'(t) = -2t\sin(2t)$ on peut prendre $K=2$. $\frac{K}{2n} \leq 10^{-2}$ pour $n=100$

$\left|I - \sum_{k=0}^{99}\cos\left((\frac{k}{n})^2\right)\right| \leq \frac{1}{100}.$

5.4: $g(t) = \varphi(x+t(\tilde{x}-x), y+t(\tilde{y}-y))$ $|g(1)-g(0)| \leq \sup|g'(t)|$ $g'(t) = \frac{\partial\varphi}{\partial x}.(\tilde{x}-x) + \frac{\partial\varphi}{\partial y}.(\tilde{y}-y)$

donc $|g'(t)| \leq K|\tilde{x}-x| + K|\tilde{y}-y| \leq \begin{pmatrix}K\\K\end{pmatrix}\cdot\begin{pmatrix}|\tilde{x}-x|\\|\tilde{y}-y|\end{pmatrix} \leq \left\|\begin{matrix}K\\K\end{matrix}\right\|\cdot\left\|\begin{matrix}|\tilde{x}-x|\\|\tilde{y}-y|\end{matrix}\right\| = \sqrt{2}\,K\,\|\vec{n\tilde{n}}\|.$

or $g(1)-g(0) = \varphi(\tilde{n}) - \varphi(n)$

5.5: $\left|\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\varphi(\frac{k}{n},\frac{l}{n}) - \int_{l/n}^{(l+1)/n}\int_{k/n}^{(k+1)/n}\varphi(x,y)\,dx\,dy\right| \leq \sum_{k,l}\int_{l/n}^{(l+1)/n}\int_{k/n}^{(k+1)/n}\underbrace{\left|\varphi(\frac{k}{n},\frac{l}{n}) - \varphi(x,y)\right|}_{\leq K|x-\frac{k}{n}|+|y-\frac{l}{n}|}dx\,dy \leq \sum_{k,l}\int_{l/n}^{(l+1)/n}\int_{k/n}^{(k+1)/n} K(x-\frac{k}{n}) + K(y-\frac{l}{n})\,dx\,dy$

$\leq \sum_{k,l}\int_{l/n}^{(l+1)/n}\left[\frac{K}{2}(x-\frac{k}{n})^2\right]_{k/n}^{(k+1)/n} + \frac{K}{n}(y-\frac{l}{n})\,dy = \sum_{k,l}\int_{l/n}^{(l+1)/n}\frac{K}{2n^2} + \frac{K}{2n}\left[(y-\frac{l}{n})^2\right]_{l/n}^{(l+1)/n} = \sum_{k,l}\frac{K}{2n^3} + \frac{K}{2n^3} = n^2\frac{K}{n^3} = \frac{K}{n}.$