$$\frac{1}{u(1+u)} = \frac{-1}{1+u} + \frac{1}{u} \qquad \int_{1/2}^{1} \frac{1}{u(u+1)}\,du = \int_{1/2}^{1} \frac{1}{u} - \frac{1}{u+1}\,du = \Big[\ln u - \ln(u+1)\Big]_{1/2}^{1} = \ln\frac{3}{2}$$

$$I_2 = \int_0^{\pi/3} \frac{\tan t}{1+\cos t}\,dt \qquad x = \cos t \quad \frac{dx}{dt} = -\sin t \qquad \begin{array}{l} t=0 \rightarrow x = \cos 0 = 1 \\ t = \frac{\pi}{6} \rightarrow x = \cos\frac{\pi}{3} = \frac{1}{2} \end{array}$$

$$= \int_0^{\pi/3} \frac{1}{\cos t\,(1+\cos t)}(\sin t)\,dt = -\int_1^{1/2} \frac{1}{u(1+u)}\,du = \ln\left(\frac{3}{2}\right)$$

Exo 2 : $\dfrac{\partial \varphi}{\partial x} = \ln(1+x^2+y^2) + \dfrac{2x^2}{1+x^2+y^2}$.

Exo 3 : $f(t) = \varphi(t, at+bt^2)$ $\qquad f'(t) = \dfrac{\partial \varphi}{\partial x}(t, at+bt^2) + (a+2bt)\dfrac{\partial \varphi}{\partial y}(t, at+bt^2)$.

3.2 : $f''(t) = \dfrac{\partial^2 \varphi}{\partial x^2}(t, at+bt^2) + 2(a+2bt)\dfrac{\partial^2 \varphi}{\partial x \partial y}(t, at+bt^2) + (a+2bt)^2\dfrac{\partial^2 \varphi}{\partial y^2}(t, at+bt^2) + 2b\dfrac{\partial \varphi}{\partial y}(t, at+bt^2)$

$f''(0) = \dfrac{\partial^2 \varphi}{\partial x^2}(0,0) + 2a\dfrac{\partial^2 \varphi}{\partial x \partial y}(0,0) + a^2\dfrac{\partial^2 \varphi}{\partial y^2}(0,0) + 2b\dfrac{\partial \varphi}{\partial y}(0,0)$

3.3 : $f'(0) = 6 + a\,2$ donc $f'(0) = 0$ ssi $a = -3$

$f''(0) = \dfrac{\partial^2 \varphi}{\partial x^2}(0,0) - 6\dfrac{\partial^2 \varphi}{\partial x \partial y}(0,0) + 9\dfrac{\partial^2 \varphi}{\partial y^2}(0,0) + 4b$.

$= 4 + 4b$ donc $f''(0) = -1$ ssi $b = -\dfrac{5}{4}$

3.4 : $f'(0) = 0$ et $f''(0) < 0$ donc f possède un max local en 0, or justement les valeurs prise par f autour de 0 sont égales aux valeurs de φ sur la parabole d'équation $y = ax+bx^2$ au voisinage de $(0,0)$

Exo 4 : 4.1 : $\dfrac{\partial \varphi}{\partial x} = \dfrac{-2}{x^2yz} + 2x \qquad \dfrac{\partial \varphi}{\partial y} = \dfrac{-2}{xy^2z} + 2y \qquad \dfrac{\partial \varphi}{\partial z} = \dfrac{-2}{xyz^2} + 2z$

4.2 : Soit x_0, y_0, z_0 un point critique $\dfrac{1}{x_0y_0z_0} = x_0^2$ et $\dfrac{1}{x_0y_0z_0} = y_0^2$ et $\dfrac{1}{x_0y_0z_0} = z_0^2$

donc $x_0^2 = y_0^2 = z_0^2$ or $x_0 > 0$ $y_0 > 0$ $z_0 > 0$ donc $x_0 = y_0 = z_0$.

4.3 : $\dfrac{1}{x_0^3} = x_0^2$ donc $x_0^5 = 1$ donc $x_0 = 1$ donc $P_0 = (1;1;1)$

$\dfrac{\partial^2 \varphi}{\partial x^2} = \dfrac{4}{x^3yz} + 2 \qquad \dfrac{\partial^2 \varphi}{\partial y^2} = \dfrac{4}{xy^3z} + 2 \qquad \dfrac{\partial^2 \varphi}{\partial z^2} = \dfrac{4}{xyz^3} + 2 \qquad \dfrac{\partial^2 \varphi}{\partial x \partial y} = \dfrac{2}{x^2y^2z}$

$\dfrac{\partial^2 \varphi}{\partial x \partial z} = \dfrac{2}{x^2yz^2} \qquad \dfrac{\partial^2 \varphi}{\partial y \partial z} = \dfrac{2}{xy^2z^2}$ $\qquad$ la hessienne en P_0 $\quad H = \begin{pmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{pmatrix} = 2\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

4.4 : $P_H(\lambda) = \begin{vmatrix} 6-\lambda & 2 & 2 \\ 2 & 6-\lambda & 2 \\ 2 & 2 & 6-\lambda \end{vmatrix} = (6-\lambda-2)^2(6-\lambda+4) = (4-\lambda)^2(10-\lambda)$

Les valeurs propres de H sont strictement positives (10 et 4) donc φ possède au point critique P_0 un minimum local strict.

5.1 : Théorème spectral pour 1 ~~forme~~ matrice symétrique. $\|P^tX\|^2 = (P^tX)^t P^tX = X^tPP^tX = X^tIX$

5.3 : $X^tSX = (P^tX)^t D (P^tX) = \lambda_1 U_1^2 + \lambda_2 U_2^2$ où $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ et $P^tX = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ $\qquad X^tSX \geqslant \lambda_1(U_1^2+U_2^2) = \lambda_1(X_1^2+X_2^2)$

5.4 : $\varphi(X) - \varphi(0) = d_0\varphi X + \frac{1}{2}d_0^2\varphi XX + o(\|X\|^2) = \frac{1}{2}(\lambda_1U_1^2 + \lambda_2U_2^2) + o(X_1^2+X_2^2) \geqslant (X_1^2+X_2^2)\underbrace{\left(\frac{\lambda_1}{2} + \varepsilon(X_1,X_2)\right)}_{>0 \text{ pour } (X_1,X_2) \text{ assez petit}}$