L2GC – – S3 – session 2 – Juin 2022

Exercice 1 : $\sin x + \cos 2x = 0 \iff \cos(\frac{\pi}{2}-x) + \cos(2x) = 0 \iff \cos(\pi + \frac{\pi}{2} - x) = \cos 2x$

$\sin x + \cos 2x = 0 \iff 3\frac{\pi}{2} - x = 2x\ [2\pi]$ ou $\frac{3\pi}{2} - x = -2x\ [2\pi]$

$\iff 3x = 3\frac{\pi}{2}\ [2\pi]$ ou $x = -\frac{3\pi}{2}\ [2\pi] \iff x = \frac{\pi}{2}\ [\frac{2\pi}{3}]$ ou $x = -\frac{3\pi}{2}\ [2\pi]$.

Les solutions sont donc $\frac{\pi}{2}$; $\frac{\pi}{2} - \frac{2\pi}{3} = \frac{-\pi}{6}$; $\frac{\pi}{2} - \frac{4\pi}{3} = \frac{-5\pi}{6}$; $-\frac{3\pi}{2} + 2\pi = \frac{\pi}{2}$.

$S = \{\frac{\pi}{2} ; -\frac{\pi}{6} ; -\frac{5\pi}{6}\}$.

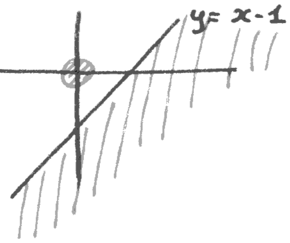
Exercice 2 : $\int_1^\infty \frac{1}{1+x^2}\,dx = [\text{Arctan } x]_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ $\quad \int_1^A \frac{1}{x} - \frac{1}{x+1}\,dx = [\ln x - \ln x+1]_1^A = \ln(\frac{A}{A+1}) + \ln 2$

$\lim_{A\to\infty} \ln(\frac{A}{A+1}) = 0$ donc $\int_1^\infty \frac{1}{x(x+1)}\,dx = \ln 2$.

$\int_0^{\sqrt{\pi}} x \sin(x^2)\,dx = [-\frac{1}{2}\cos(x^2)]_0^{\sqrt{\pi}} = -\frac{1}{2}\cos\pi + \frac{1}{2}\cos(0) = 1$.

Exercice 3 : Pour que la formule ait un sens $y - x + 1 \geq 0$ et $x^2 + y^2 \neq 0$.



$$\frac{\partial \varphi}{\partial y}(xy) = \frac{x\frac{(x^2+y^2)}{2\sqrt{y-x+1}} - 2yx\sqrt{y-x+1}}{(x^2+y^2)^2}$$

$$= \frac{x^3 + xy^2 - 4xy(y-x+1)}{2\sqrt{y-x+1}\,(x^2+y^2)^2} = \frac{x^3 - 3xy^2 + 4x^2y - 4xy}{2\sqrt{y-x+1}\,(x^2+y^2)^2}$$

Exercice 4 : $\frac{\partial \varphi}{\partial x} = y - \frac{1}{x^2}$ $\quad \frac{\partial \varphi}{\partial y} = x - \frac{1}{y^2}$ $\quad \begin{cases} \frac{\partial \varphi}{\partial x} = 0 \\ \frac{\partial \varphi}{\partial y} = 0 \end{cases} \iff \begin{cases} xy - \frac{1}{x} = 0 \\ xy - \frac{1}{y} = 0 \end{cases} \iff \begin{cases} \frac{1}{x} = \frac{1}{y} \\ x^2 - \frac{1}{x} = 0 \end{cases}$

$x^2 - \frac{1}{x} = 0 \iff x^3 - 1 = 0 \iff (x-1)(x^2+x+1) = 0 \iff x = 1$.

Donc $P = (1;1)$.

4.2 : $\frac{\partial^2 \varphi}{\partial x^2} = \frac{2}{x^3}$ $\quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{2}{y^3}$ $\quad \frac{\partial^2 \varphi}{\partial x \partial y} = 1$ $\quad H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

4.3 : $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3$ $\quad \Delta = 16 - 12 = 4 = 2^2$

$\lambda = \frac{4 \pm 2}{2}$ • $\begin{cases} \lambda_1 = 3 \\ \lambda_2 = 1 \end{cases}$

4.4 : Les 2 V.P. sont strict positives donc φ possède en P un min local strict

4.5 : $\frac{\partial \varphi}{\partial x} = yz - \frac{1}{x^2}$ $\quad \frac{\partial \varphi}{\partial y} = xz - \frac{1}{y^2}$ $\quad \frac{\partial \varphi}{\partial z} = xy - \frac{1}{z^2}$

En un point fixe $xyz = \frac{1}{x}$ et $xyz = \frac{1}{y}$ et $xyz = \frac{1}{z}$ donc $x = y = z$.

donc $x^4 = 1$, il y a alors 2 points fixe $P_1 = (1,1,1)$ et $P_2 = (-1,-1,-1)$

Determinons les Hessiennes en P_1 et en P_2 $\quad H_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ $\quad H_2 = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} = -H_1$.

$\frac{\partial^2 \varphi}{\partial x^2} = \frac{2}{x^3}$ $\quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{2}{y^3}$ $\quad \frac{\partial^2 \varphi}{\partial z^2} = \frac{2}{z^3}$ $\quad \frac{\partial^2 \varphi}{\partial x \partial y} = z$ $\quad \frac{\partial^2 \varphi}{\partial x \partial y} = z$ $\quad \frac{\partial^2 \varphi}{\partial y \partial z} = x$

$P_{H_1}(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 + 1 + 1 - (2-\lambda) - (2-\lambda) - (2-\lambda) = 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 4 + 3\lambda = -\lambda^3 + 6\lambda^2 - 9\lambda + 4$

$P_{H_1}(\lambda) = (\lambda - 1)(-\lambda^2 + 5\lambda - 4) = (\lambda-1)^2(-\lambda+5)$

Les valeurs propres de H_1 sont 1 ; 1 et 5 donc φ possède 1 min local strict en P_1.

―― H_2 ―― -1 ; -1 et -5 ―― max ―― P_2.