L1-MPI-S2 - 2009/2010 - Examen du 19 mai 2010 - SESSION 1

Examination of Mathematics for Sciences (MS2)

Duration: **3** hours

Documents, calculators and mobile phones are forbidden

Exercice 1 : **a)** Calculate a primitive of the application $y \mapsto y \ln(y^2 + 1)$.
b) Let $I = \int_2^3 x \ln[(x-2)^2 + 1] \, \mathrm{d}x$. Put $y = x - 2$, and show that $I = I_1 + 2I_2$ where

$$I_1 = \int_0^1 y \ln(y^2 + 1) \, \mathrm{d}y \quad \text{et} \quad I_2 = \int_0^1 \ln(y^2 + 1) \, \mathrm{d}y.$$

c) Compute I_2 by using integration by parts.
d) Deduce the value of I.

Exercice 2 : Our aim in this exercise is to compute the integrals I, J, K below.
1.a) Let $P(t) = t^3 + t^2 + t + 1$. Factorize this polynomial on $\mathbb{R}$.
(Hint : one of his roots is the complex number i).
1.b) Deduce the decomposition of the rational fraction $f(t) = \dfrac{t}{P(t)}$
into simple elements.
1.c) Deduce the value of the integral $I = \displaystyle\int_0^1 f(t) \, \mathrm{d}t$
2.a) Write a formula which links $\sin t$ with $\cos(2t)$.
2.b) Calculate the integral $J = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(2t)}{1 + \sin^2 t} \, \mathrm{d}t$.
3.a) Give the domain of definition of the application $x \mapsto \dfrac{1}{\sqrt{3-x}}$ and the domain of definition of one of its primitives (give a brief justification), then calculate the expression of such a primitive.
3.b) Deduce (by using a change of variable inspired from (2.a)) the value of the integral

$$K = \int_0^{\frac{\pi}{2}} \frac{\sin(2t)}{\sqrt{1 + \sin^2 t}} \, \mathrm{d}t.$$

Exercice 3 : We propose to solve the following differential equation :

$$y' \sin x - y \cos x = 2x \sin^2 x. \qquad (*)$$

on the inteval $I =]0, \pi[$.
1) Justify why equation $(*)$ is equivalent to

$$y' - y \operatorname{cotan} x = 2x \sin x. \qquad \text{(E)}$$

on the interval I.
2) On interval I, solve the homegeneous differential equation associated to (E) ,that is

$$y' - y \operatorname{cotan} x = 0. \qquad (\mathrm{E}_0)$$

Give the set of solutions of (E_0).
3) Find a particular solution of equation (E).
4) Deduce the set of solutions of (E).

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Exercice 4 : Let f be the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = e^{xy}$.

1) Compute the partial derivatives of f (that we will denote by $\partial_x f \equiv \frac{\partial f}{\partial x}$ and $\partial_y f \equiv \frac{\partial f}{\partial y}$).

2) Show that f has a unique critical point (x_0, y_0) to be precised.

3) Compute the second order partial derivatives $\partial_{xx} f, \partial_{yx} f, \partial_{xy} f$ and $\partial_{yy} f$ at an arbitrary point (x, y).

4) Calculate the value of the following determinant, at the critical point (x_0, y_0) found in the preceding question, that is :

$$\Delta(x_0, y_0) = \det \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2}(x_0, y_0) & \dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0) & \dfrac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{pmatrix}.$$

5) Deduce from this, if the critical point corresponds to a local extremum (min or max) or to a saddle point.

Exercice 5 : We propose to find all functions $f : \mathbb{R} \to \mathbb{R}$ which are derivable and such that

$$(*) \qquad \forall x \in \mathbb{R}, \quad 2f'(x) + 3f(-x) = x^2 + 1.$$

1) Justify why can we differentiate the two members of $(*)$. After differentiating, we will get for all $x \in \mathbb{R}$ an equality that we will denote by $(**)$.

2) By using $(*)$ and $(**)$, find one second order differential equation, that we will denote by (E), with unknown function $y = f(x)$.

3) Solve the associated homogeneous equation (E_0) associated to (E).

4) Find a particular solution of (E) and then write the set of solutions of (E).

5) Put the obtained result in $(*)$ to find the set of solutions f that we were looking for.