L1 - Bilingual Group - MS2 - 2008/2009 - **20 May 2009** - Session 1

Examination in Mathematics for Sciences (MS2)

Duration: **3** hours – Documents and any type of calculators are not allowed

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Exercice 1 : 1. Decompose into simple elements on $\mathbb{C}$, after that on $\mathbb{R}$, the rational fraction $\dfrac{1}{X(X^2+1)}$.

2. Compute the value of the integral : $I = \displaystyle\int_1^{\mathrm{e}} \frac{x}{(x^2+1)^2} \ln x \,\mathrm{d}x$. (Hint : first, integration by parts)

Exercice 2 : 1. Calculate $I = \displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$.

2. Let $f(x) = \dfrac{x\sin x}{1+\cos^2 x}$. Show the identity : $f(x) + f(\pi - x) = \dfrac{\pi \sin x}{1+\cos^2 x}$.
3. Deduce from the previous questions the value of $J = \displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$.
4. By computing J directly by integration by parts, infer the value of the integral $K = \displaystyle\int_0^{\pi} \mathrm{Arctan}(\cos x)\,\mathrm{d}x$.

Exercice 3 : Let $a, b \in \mathbb{R}$, $a < b$. Let $I_{m,n} = \displaystyle\int_a^b (x-a)^m (b-x)^n \,\mathrm{d}x$, with $(m,n) \in \mathbb{N}^2$.

1. Show that $I_{m,n} = \dfrac{m}{n+1} I_{m-1,n+1}$, $\forall (m,n) \in \mathbb{N}^* \times \mathbb{N}$ and $I_{m,n} = -\dfrac{n}{m+1} I_{m+1,n-1}$, $\forall (m,n) \in \mathbb{N} \times \mathbb{N}^*$.
2. Compute $I_{0,m+n}$ and deduce from the previous questions that $I_{m,n} = \dfrac{m!n!(b-a)^{m+n+1}}{(m+n+1)!}$.

Exercice 4 : Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^3(1-4y^2) + y^2$.

1. Compute its first order partial derivatives (denoted as $\partial_x f \equiv \frac{\partial f}{\partial x}$ and $\partial_y f \equiv \frac{\partial f}{\partial y}$).
2. Show that f has only 3 critical points, namely $(0,0)$, $(\frac{1}{\sqrt[3]{4}}, \frac{1}{2})$ and $(\frac{1}{\sqrt[3]{4}}, -\frac{1}{2})$.
3. Compute in each point $(x,y) \in \mathbb{R}$ the second order partial derivatives of f (denoted by $\partial_{xx} f \equiv \frac{\partial^2 f}{\partial x^2}$, $\partial_{xy} f \equiv \frac{\partial^2 f}{\partial x \partial y}$, $\partial_{yx} f \equiv \frac{\partial^2 f}{\partial y \partial x}$, $\partial_{yy} f \equiv \frac{\partial^2 f}{\partial y^2}$).
4. Compute for each $(x,y) \in \mathbb{R}$, the expression of the determinant $\det H(x,y) = \begin{vmatrix} \partial_{xx} f(x,y) & \partial_{xy} f(x,y) \\ \partial_{yx} f(x,y) & \partial_{yy} f(x,y) \end{vmatrix}$.
5. Give the values of $\det H(x_i, y_i)$ where (x_i, y_i), $i = 1,2,3$, are the critical points of f.
6. According to the values found at the preceding question, discuss and decide (if possible) on the nature of the critical points (extremum, saddle points ?).
7. Calculate $f(0,0)$. Write the expression of $f(x,y)$ in polar coordinates and use it in order to study the sign of f in an arbitrarily small ball centered on $(0,0)$. Conclude : is $f(0,0)$ or not a local extremum of f ?

Exercice 5 : Let $f : I \to \mathbb{R}$ (with I interval of $\mathbb{R}$) be a differentiable function and let F be a primitive of f. Let $(\mathbf{E})$ be the differential equation (with unknown function $y : I \to \mathbb{R}$) : $y'' + f\,y' + f'\,y = 0$.

1. Show that $(\mathbf{E})$ is equivalent to the sentence : there is some $C \in \mathbb{R}$ such that $(\mathbf{e}_C)$: $y' + fy = C$ on I.
2. Denote by $(\mathbf{e}_C^{'})$ the homogenous (null right hand side) equation associated to $(\mathbf{e}_C)$. Give the expression of the general solution y_0 of $(\mathbf{e}_C^{'})$ (as an expression depending on F).
3. Use the variation of the constant method in order to find an expression for a particular solution y_{p} of $(\mathbf{e}_C)$ (as an expression depending on F). Deduce from the previous questions the general solution of $(\mathbf{e}_C)$.
4. Give the general solution of $(\mathbf{E})$ (as an expression depending on F).
5. For $x \in I = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ let $f(x) = \dfrac{\tan x}{3+\cos(2x)}$. Compute a primitive F of f.

6. Deduce, for this case, the general solution of $(\mathbf{E})$ in terms of a primitive G of $g(x) = \sqrt[4]{2 + \tan^2 x}$ (one shall not try to compute G explicitely).

Note : *In the exercises 4 and 5 the last question is lightly harder than the others.*