Final examen of Mathematics for Sciences (MS1)

14th January 2012 – **Duration : 2h30**

Documents and electronic devices are forbidden.

Remark: Some questions which have a higher degree of difficulty have been reported with an asterisk.

Exercice 1. (3 points) Let

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \end{pmatrix}.$$

1. Compute the product $P = AB$.

2. Evaluate its determinant $\det P$.

3. Is the matrix P invertible?

Exercice 2. (4 points)

1. Find the Taylor expansions of order 4 in zero of the following functions

$$f(x) = \ln(1 + x^2), \quad g(x) = \sin(2x) \cdot \cos(x).$$

2. Evaluate $\lim_{x \to 1} \frac{\ln(x^2) - \sin(x^2 - 1)}{(x^2 - 1)^2}$.

(*Hint: set* $u(x) = x^2 - 1$ *and write* $\ln(x^2) = \ln((x^2 - 1) + 1)$.)

Exercice 3. (6 points) Let $\mathcal{L}_1$ be the line with equation $3x + 4y - 10 = 0$ and let $\mathcal{L}_2$ be the line through the point $S = S(1; 2)$ orthogonal to $\vec{n} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

1. Find an equation for $\mathcal{L}_2$.

2. Find the point of intersection of the lines $\mathcal{L}_1$ and $\mathcal{L}_2$. We denote it by Q.

3. Consider a point $B(a, b)$ where $a, b \in \mathbb{R}$ are two parameters.

(3.a) Determine the orthogonal projections of B onto $\mathcal{L}_1$ and $\mathcal{L}_2$ that we denote by P_1 and P_2 respectively.

(3.b) Find the distances from B to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(3.c)* Give a necessary and sufficient condition for the coordinates a and b such that (QB) is the angle bisector of the angles formed by $\mathcal{L}_1$ and $\mathcal{L}_2$.
(*Hint: make a figure and see the definition at the end.*)

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Exercice 4. (7 points) Consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ verifying

$$\text{(R)} \quad \forall n \in \mathbb{N}^*, \quad u_{n+1} - 3u_n = 1.$$

1. **(a)** Suppose that $(u_n)_{n\in\mathbb{N}^*}$ converges to a limit l for $n \to \infty$. Find l.

(b) Determine a function f such that $u_{n+1} = f(u_n)$ for all $n \in \mathbb{N}^*$. Give the fixed points of f.

(c) Draw the cobweb diagram (spiral process) associated to $(u_n)_{n\in\mathbb{N}^*}$ where $u_1 = 1$.

(d) Deduce (without proof) if such sequence converges to l found in question (1.a).

2.* **(a)** Let $(v_n)_{n\in\mathbb{N}^*}$ be the sequence given by the following recursive relation

$$(\text{R}_0) \quad \forall n \in \mathbb{N}^*, \quad v_{n+1} - 3v_n = 0.$$

Show that $(v_n)_{n\in\mathbb{N}^*}$ is a geometric sequence with common ratio 3. Give its explicit formula.

(b) Set $p_n = \lambda_n v_n$ for $n \in \mathbb{N}^*$ where $(v_n)_{n\in\mathbb{N}^*}$ satisfies (R_0) with $v_1 = 1$. Prove that the sequence $(p_n)_{n\in\mathbb{N}^*}$ verifies (R) if and only if $\lambda_{n+1} - \lambda_n = 3^{-n}$ for all $n \in \mathbb{N}^*$.

(c) Prove that λ_n is the sum of $n-1$ first terms of a geometric sequence with common ratio $\frac{1}{3}$ et initial term $\frac{1}{3}$. Compute explicitly λ_n. Deduce an expression of p_n.

(d) Deduce from (2.a) and (2.c) the expression of all sequences $(u_n)_{n\in\mathbb{N}^*}$ verifying (R). Deduce that the constant sequence $-\frac{1}{2}$ satisfies obviousely (R).

(e) For each l found in (1.a), precise what are sequences among those satisfying (R) converge to l.

Definition: *The angle bisector is the line that divides the angle into two angles with equal measures. Each point of an angle bisector is equidistant from the sides of the angle.*