L1 MI-PC
2011/2012
Mathematics 1

Examination, 20th of January 2012

Duration: 2h30mn. All documents, Calculators, Translators and mobile phones are forbidden !

Exercise 1. Let $z_0 = -4\sqrt{3} + 4i$.
a) Write z_0 into an exponential form.
b) Solve in $\mathbb{C}$ the equation $z^3 = z_0$. Give the solutions in their exponential form.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ and $l \in \mathbb{R}$. Give the mathematical definition of $\lim\limits_{x\to+\infty} f(x) = l$.

Exercise 3.
a) Recall (without justification) the following limits: $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$ and $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x}$.
b) Study the following limit : $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{\sin(2x)}$.

Exercise 4. Let f be the function defined by $f(x) = \dfrac{\sqrt{x+4}-2}{x}$.
a) Show that f is defined and continuous on $[-4, 0[\cup]0, +\infty[$.
b) Show that f is differentiable on $]-4, 0[\cup]0, +\infty[$. Calculate $f'(x)$.
c) Show that f can be extended by continuity at point 0. Write the obtained continuous extension of f.

Exercise 5. Let $(u_n)_{n\in\mathbb{N}^*}$ be the sequence defined by recursion by: $u_1 = 1$ and, for all $n \in \mathbb{N}^*$, $u_{n+1} = u_n + \dfrac{1}{n+1}$, and let $(v_n)_{n\in\mathbb{N}^*}$ et $(w_n)_{n\in\mathbb{N}^*}$ be the sequences defined by

$$v_n = u_n - \ln(n) \text{ and } w_n = u_n - \ln(n+1).$$

a) Recall the Mean Value Theorem. Application : Let $n \in \mathbb{N}^*$, show that

$$\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}.$$

b) Deduce that, for all $n \in \mathbb{N}^*$, $\ln(n+2) - \ln(n+1) < \dfrac{1}{n+1}$.
c) Show that for all $n \in \mathbb{N}^*$, $\ln(n+1) < u_n$. (We can use a mathematical induction reasoning.)
d) Determine the limit of the sequence $(u_n)_n$ as $n \to +\infty$.
e) Show that the sequence $(v_n)_n$ is strictly decreasing. Indication : use question a).
f) Show that for all $n \in \mathbb{N}^*$, $v_n > 0$. Deduce from this that $(v_n)_n$ is convergent. We denote by γ its limit.
g) By using the same ideas as in questions e) and f), show that the sequence $(w_n)_n$ is convergent. What is its limit ?
h) Find explicit constants m, M such that $m < \gamma < M$ and $M - m < \frac{1}{10}$.