Université de Cergy-Pontoise
UFR de Sciences et Techniques

Licence L1, 2010/2011
Monday, 10th of january 2011

Examination of MATHEMATICS (M1)

(Duration : 2 hours and 30 minutes - Documents, calculators and mobile phones are forbidden)

Exercice 1 : Let p and n be two elements of $\mathbb{N}$ such that $p \leqslant n$. Give the definition of the binomial coefficient $\begin{pmatrix} n \\ p \end{pmatrix}$. Show that the integer number $n(n-1)\cdots(n-p+1)$ is a multiple of $p\,!$

Exercise 2 : Let u_n be a real sequence and $l \in \mathbb{R}$. Give the mathematical definition of

$$u_n \longrightarrow \ell \text{ when } n \longrightarrow +\infty.$$

Exercise 3 :

1) Let $A = 1 - i$. Calculate A^2 and A^4.
2) Solve the equation $z^2 - 2z + 1 + \dfrac{1}{2}i = 0 \quad (z \in \mathbb{C})$.

Exercise 4 : Calculate the limits of the following quantities if they exist

$$f(x) = \frac{x}{\sin(3x)} \text{ when } x \longrightarrow 0, \qquad g(x) = \frac{3 + \ln x}{5 + 2\ln x} \text{ when } x \longrightarrow +\infty.$$

Exercise 5 :

1) Let $\theta \in \mathbb{R}$. Show that

$$\frac{d}{dx}\left(\sin(x+\theta)\right) = \sin\left(x + \theta + \frac{\pi}{2}\right)$$

and then show [1] that for all $n \in \mathbb{N}^*$ we have for the nth derivative :

$$\frac{d^n}{dx^n}(\sin x) = \sin\left(x + n\frac{\pi}{2}\right).$$

2) Let f be a function which is defined on $\mathbb{R}$ by $f(x) = x^2 \sin x$. Show that $\forall n \in \mathbb{N}^*$ we have

$$f^{(n)}(x) = x^2 \sin\left(x + n\frac{\pi}{2}\right) + A_n x \sin\left(x + (n-1)\frac{\pi}{2}\right) + B_n \sin\left(x + (n-2)\frac{\pi}{2}\right)$$

where A_n and B_n are constants to be precised (one can use Leibnitz's formula).

Exercise 6 : Let φ a function defined on $]-1,1[$ by

$$\varphi(t) = \frac{t}{1-t^2}.$$

1) Calculate

$$\lim_{\substack{t \longrightarrow 1 \\ t<1}} \varphi(t) \qquad \text{and} \qquad \lim_{\substack{t \longrightarrow -1 \\ t>-1}} \varphi(t).$$

2) Let f be a function which is derivable on $\mathbb{R}$ such that $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = \ell$ with $\ell \in \mathbb{R}$.
We consider a new function defined on $[-1,1]$ by

$$u(t) = \begin{cases} f\left(\varphi(t)\right) & \text{if} \quad t \in]-1,1[\\ \ell & \text{if} \quad t = -1 \text{ or } t = 1 \end{cases}.$$

a) Show that u is continuous on $[-1,1]$ and derivable on $]-1,1[$. Show that there exists $t_0 \in]-1,1[$ such that $f'\left(\varphi(t_0)\right) \times \varphi'(t_0) = 0$ (one can use Rolle's theorem).

b) Deduce that there exists $x_0 \in]-\infty, +\infty[$ such that $f'(x_0) = 0$.

[1] When f is a n times derivable function $\frac{d^n f(x)}{dx^n}$ denotes $f^{(n)}(x)$.